EXACT SOLUTIONS FOR NONCLASSICAL STEFAN-LIKE PROBLEMS USING THE REFINED INTEGRAL METHOD

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RESUME :
An efficient analytical approach in solving nonlinear 1-D transient diffusion problems is presented, in case of a prescribed initial distribution of the diffused property. The solution is based on the refined integral method where the kernel function is predicted according to the initial distribution. It is shown that the exact solution can be recovered in such case as claimed by Sadoun *et al.*[1]. Four benchmark test examples, including linear and nonlinear time dependence of the free boundary, were conducted; each leading to the exact solution through the refined integral method. The main results of this investigation seem to go against some misleading statements raised up in the literature [2-4].

Mots-clés : Exact solution, Stefan problem, Inverse Stefan problem, Source, Sink, Refined integral method, RIM

I. GENERAL SETTING

The present paper deals with 1-D nonlinear transient diffusion problem with or without source term. The problem, referred to as Stefan problem, includes a moving boundary making it non-linear. More explicitly, the problem is described as follows:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad (1)$$

where u is the dependent variable and will be specified at a later stage, $f(x,t)$ is the forcing term, t and x refer to time and space coordinates respectively. $s(t)$ stands for the moving interface position and α for the diffusivity. The initial conditions are given by :

$$s(t) = s_0 \qquad (2)$$

and

$$u = \begin{cases} \phi(x), & 0 < x < s_0 \\ u_i, & x \geq s_0 \end{cases} \qquad (3)$$

Otherwise, the space domain is initially divided into two regions: i.e ranging from $x = 0$ to $x = s_0$ with a prescribed distribution $\phi(x)$, of the dependent variable, while the second being for $x > s_0$ with a uniform property, equals to its initial value u_i. $f(x,t)$ and $\phi(x)$ are assumed to be analytical. This setting includes some moving boundary problems with important applications since the diffusion may concern heat [5] as well as species [6,7]. In the following, to alleviate the presentation, we arbitrary restrict the formulation to the heat diffusion so that u will stand for temperature. Generally, for such problems, two conditions are specified on the moving boundary, $s(t)$ as follows :

$$u|_s = u_i \qquad (4)$$

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$$-k\left.\frac{\partial u}{\partial x}\right|_s = \kappa \frac{ds}{dt} \qquad (5)$$

where k and κ refer respectively to the conductivity and the latent heat per unit volume associated to the phase change process. As outlined by Johansson *et al.* [8] both conditions (Eqs.4,5) can be replaced by the more general boundary conditions :

$$u|_s = h_1(t) \qquad (6)$$

$$\left.\frac{\partial u}{\partial x}\right|_s = h_2(t) \qquad (7)$$

The condition closing the mathematical model depends on the problem under consideration. It may be defined among the function prescribing the time-evolution of the moving boundary, $s(t)$, such as in the inverse Stefan problems, or the temperature or the heat flux applied on the fixed boundary $x=0$. In the latter, we will be concerned only by the following condition :

$$u|_0 = u_0 \qquad (8)$$

In this paper, the temperature distribution as well as the moving boundary position, when being a part of the solution, are investigated. For this purpose, our approach is based on the use the refined integral method which does not need any linearization of the required equations and appears to be particularly adequate for a fixed boundary condition of the first kind (Eq.8). Indeed, in such case, the analysis is proved to be simpler and led to more accurate solutions. In the following, the solutions of four-benchmark tests of Stefan-like problems are investigated. Three of them concern the inverse Stefan problems consisting on finding the temperature $u(x,t)$ satisfying Eq.(1) without a source term, $f(x,t)$, but having a known moving boundary position $s(t)$. The last example considers a Stefan problem with a forcing term and consists on finding the time-evolution of the moving boundary position $s(t)$ and recovering, then, the temperature distribution $u(x,t)$ in the active layer $[0,s(t)]$.

It should be observed that according to Gupta and Banik [2] as well as, more recently, Mitchell [4] conventional integral methods do not work for non-uniform initial distribution whereas Myers and Mitchell [9] state that these methods will produce acceptable approximate solutions, only for small time, as long as the applied condition, at the surface exchange boundary, varies exponentially with time. The present work highlights the fact that, in general, such statements cannot be verified; moreover, conventional methods, in some cases, can even provide the exact solution if the exact distribution may be described by as many time-dependent parameters as the number of specified boundary conditions.

II. INTEGRAL FORMULATION

The refined integral method formulated in a moving domain, as developed in [10,11], combines the heat balance integral and the double integral methods. The objective being to remove the term involving the derivative of the unknown distribution when the heat flux at the fixed boundary is not specified by the mathematical model. To that aim, the heat conduction equation is first approached by an overall energy balance in the layer $[0,s(t)]$:

$$\frac{d}{dt}\int_0^s u dx - u|_s \frac{ds}{dt} - \alpha\left.\frac{\partial u}{\partial x}\right|_s - \int_0^s f dx = -\alpha\left.\frac{\partial u}{\partial x}\right|_0 \qquad (9)$$

The result is referred to as the Goodman heat balance integral equation in reference to his pioneering study [12]. Next, the heat diffusion equation is integrated twice with respect to space. The first integration is conducted from $x'=0$ to x' while the second from $x=0$ to $s(t)$ giving rise to the following double integral equation :

$$s\frac{d}{dt}\int_0^s u dx - \frac{d}{dt}\int_0^s xu dx - \alpha\, u|_s - \int_0^s\int_0^{x'} f dx dx' = -\alpha\, u|_0 - \alpha s\left.\frac{\partial u}{\partial x}\right|_0 \qquad (10)$$

where x' is a dummy variable. Eq.(10) involves the values at the fixed boundary $x=0$ of both the function and its derivative with respect to space while no more than one value is usually specified by the boundary conditions. Finally, upon combining these two integral equations (9,10), the refined integral equation is generated

$$\frac{d}{dt}\int_0^s xudx+\alpha\left(u|_s-u|_0\right)-\int_0^s xfdx = s\left(\alpha\frac{\partial u}{\partial x}\bigg|_s+u|_s\frac{ds}{dt}\right) \quad (11)$$

It is important to observe that the final expression of the solution is given in terms of the integral of the unknown temperature distribution u and the boundary conditions of the problem (see ref. [11]). It should be underlined that, the previous equation may also be derived by considering either the first order moment with respect to the variable space of Eq. (1) [2] or the double integration of Eq.(1) with respect to the space variable where, this time, the first integration is conducted from $x'=s$ to x and the second from $x=s$ and $x=0$ [13].

The next step in the procedure of the refined integral method is to assume a functional form which approximates the temperature profile in the active layer $x\in[0,s(t)]$. This approximate profile constitutes the key part of the refined integral procedure whereas the latter being less sensitive to the profile's form, than other integral approaches, for reasons outlined previously. The improvement introduced by the procedure developed by Sadoun *et al.* [10,11] consists on enforcing the assumed profile to verify the Stefan condition (Eq.5) instead of the alternative Stefan condition (see [1]). Enforcing both boundary conditions simultaneously improves more the accuracy of the solution [11,14]. It should be recalled that work, prior to the study of Sadoun and Si-Ahmed [10], used the latter condition since it simplifies, consequently, the analysis leading, in this case, to a first order ODE instead of a second order ODE where the integration requires the unknown initial moving front velocity. However, as shown in [1], the use of alternative Stefan condition has, on the overall, a decreasing effect on the accuracy of the solution. It has been shown [11], despite the use of the Stefan condition, the analysis may also lead to a first order ODE avoiding then the non-closure of the mathematical model.

III. EXAMPLES

To illustrate the integral solution procedure outlined in the previous section, some benchmark test examples will be considered in this section. Three of them con-cern inverse Stefan problems where the location-time history of the moving boundary position is known and include linear and non-linear time-variation of $s(t)$. In the last example, the time-evolution of $s(t)$ constitutes a part of the solution and must be determined to recover the temperature distribution in the whole layer $[0,s(t)]$.

III.1 Example 1

First, we consider the inverse one-phase Stefan problem solved by Slota [15] and Hetmaniok *et al.* [16] using the homotopy perturbation method. In this example, the authors consider $\alpha=1$, $k=1$, $\kappa=1$ and $u_i=1$. The moving boundary position and the initial temperature distribution are described respectively by:

$$s(t)=1+\frac{t}{10} \quad (12)$$

$$\phi(x)=e^{1-x} \quad (13)$$

Note that, in this example, no forcing term acts on the growth layer and a linear variation, with respect to time, of the moving boundary s is involved. The exact solution of this problem is expressed as:

$$\phi(x)=e^{1-x+t/10} \quad (14)$$

It should be underlined that, with the method previously reported [15,16], the reconstruction of the exact solution requires tedious mathematical effort since the analysis leads to solutions given by series whose convergence radius are infinite. Otherwise, for truncated series, the solution is valid only for small times. This is not the case with the present method which is based on the given initial profile $\phi(x)$ in developing a simple form for the approximate distribution $u(x,t)$. Indeed, according to the initial condition (Eq.13) of the present example, the following temperature distribution is assumed:

$$u(x,t)=\varsigma_1 e^{\varsigma_2\frac{x}{s}}+\varsigma_3 \quad (15)$$

The parameters ς_i (i=1,2,3) may be time-dependent and their initial values are obtained by identifying $u(x,0)$ with the initial distribution (Eq. 13). Taking into account the initial location of the moving boundary, $s(0)=1$ from Eq. (12), the following results hold:

$$\varsigma_1(0)=e^1,\quad \varsigma_2(0)=-1,\quad \varsigma_3(0)=0 \tag{16}$$

Substitution Eq.(15) into the conditions at the moving boundary (Eqs. 4,5) establishes the following functions relating the parameters ς_1 and ς_3 to ς_2.

$$\varsigma_1=-\frac{k}{\kappa}\frac{s}{\varsigma_2 e^{\varsigma_2}}\frac{ds}{dt},\quad \varsigma_3=u_i+\frac{k}{\kappa}\frac{s}{\varsigma_2}\frac{ds}{dt} \tag{17}$$

Finally, taking into account these relations, the expression of the moving boundary $s(t)$ and the numerical values of α, k, κ and u_i, the substitution of the resulting assumed profile in the refined integral equation (11) leads to a first order ODE :

$$\frac{d\varsigma_2}{dt}=\frac{\varsigma_2}{\psi^2}\left(\frac{20\varsigma_2^2-6\psi}{(6-(4-\varsigma_2)\varsigma_2)e^{\varsigma_2}-2(3+\varsigma_2)}+\frac{(6\psi-(6\psi-3\psi+20(\varsigma_2-1))\varsigma_2)\varsigma_2 e^{\varsigma_2}}{(6-(4-\varsigma_2)\varsigma_2)e^{\varsigma_2}-2(3+\varsigma_2)}\right) \tag{18}$$

where $\psi=10+t$. Note that the integration of Eq. (18) is not obvious analytically. The numerical integration according to the initial condition $\varsigma_2(0)=-1$ (second of Eqs.(16)) shows a solution of $\varsigma_2(t)=-1-t/10$. It results $\varsigma_1(t)=e^{1+t/10}$ and $\varsigma_3(t)=0$; consequently the profile (15) becomes $u(x,t)=e^{1-x+t/10}$ which is the exact solution of the considered example.

An alternative procedure consists on formulating the problem in terms of the parameter ς_3 for which the boundary conditions lead to the following expressions for the remaining parameters ς_1 and ς_2:

$$\varsigma_1=e^{\frac{\kappa}{k}\frac{s}{u_i-\varsigma_3}\frac{ds}{dt}},\quad \varsigma_2=-\frac{\kappa}{k}\frac{s}{u_i-\varsigma_3}\frac{ds}{dt} \tag{19}$$

Introducing the approximate profile (15), with the use of Eqs. (19), into the refined integral equation (11) to obtain an ODE for ς_3. That is:

$$\frac{d\varsigma_3}{dt}=\frac{\varsigma_3}{\psi-30(1-\varsigma_3)-\frac{3}{2}\frac{\psi^2}{\psi+10(1-\varsigma_3)\left(1-e^{\frac{1}{10(1-\varsigma_3)}}\right)}} \tag{20}$$

with the initial condition (third of Eqs. 16). It can be shown accordingly that the result yields to $\varsigma_3(t)=0$. Finally, the latter introduced into the expressions of ς_1 and ς_2 (Eqs.19) allows recovering the exact solution (Eq. 14).

III.2 Example 2

The second test example concerns the inverse Stefan problem investigated by Murio [17] numerically using the molification method, by Slota [15] applying the homotopy perturbation method, and more recently, by Johansson *et al.* [8] using what they called the method of fundamental solutions and by Liu [18] using the Lie-group shooting method. In these studies the parameters were set to $\alpha=1$, $k=1$, $\kappa=1$ and $u_i=0$ and the following functions were imposed for $s(t)$ and $\phi(x)$:

$$s(t)=\frac{\sqrt{2}}{2}\left(t+2-\sqrt{2}\right) \tag{21}$$

$$\phi(x)=e^{1-\frac{\sqrt{2}}{2}(1+x)}-1 \tag{22}$$

This test example involves also a time-linear variation of the moving boundary $s(t)$. The exact solution is :

$$u(x,t)=e^{1-\frac{\sqrt{2}}{2}(1+x)+\frac{t}{2}}-1 \tag{23}$$

It is worth underlying that Slota [15] and Johansson *et al.* [8] restricted their solutions to $t=2$ and $t=1$, respectively. That is because the series used become unbounded for large times. Obtaining accurate solutions for long times require to consider more terms in the series leading then to tedious mathematical effort. Hereafter and likewise the previous example, the shape of the initial distribution (Eq. 22) is used to choose the approximate profile $u(x,t)$. In this case, the profile (Eq. 15) used previously, is suggested. Identifying then $u(x,0)$ from Eq.(15) with the given initial condition (Eq. 22) yields :

$$\varsigma_1(0)=e^{1-\frac{\sqrt{2}}{2}},\quad \varsigma_2(0)=\frac{\sqrt{2}}{2}-1,\quad \varsigma_3(0)=0 \qquad (24)$$

Taking into account the established expressions fot ς_1 and ς_3 in term of ς_2 (Eqs. (17) obtained from Eqs.(4,5)), the profile reads :

$$u=u_i+\frac{\kappa s}{k\varsigma_2}\left(1-e^{\varsigma_2\left(\frac{x}{s}-1\right)}\right) \qquad (25)$$

Finally, after substituting Eq.(25) into the integral equation (11), the following ODE is obtained :

$$\frac{d\varsigma_2}{dt}=\frac{\varsigma_2}{\psi^2}\left(\frac{6(e^{\varsigma_2}-1)}{(6+(\varsigma_2-4)\varsigma_2)e^{\varsigma_2}-2(3+\varsigma_2)}+\frac{(4\varsigma_2+(6\psi+(3\psi+4(\varsigma_2-1))\varsigma_2)e^{\varsigma_2}}{(6+(\varsigma_2-4)\varsigma_2)e^{\varsigma_2}-2(3+\varsigma_2)}\right) \qquad (26)$$

where Eq.(21) as well as the numerical values of α, k, κ and u_i are taking into account with $\psi=t+2-\sqrt{2}$. Once again, the numerical integration of ODE (26) according to the initial condition (second of Eqs.(24)), yields $\varsigma_2=-\left(t+2-\sqrt{2}\right)/2$ and consequently $\varsigma_1=e^{\left(t+2-\sqrt{2}\right)/2}$ and $\varsigma_3=-1$. This means that the assumed profile (Eq.25) fits perfectly the exact temperature distribution (Eq.23) of the test problem.

As pointed out, in the previous example, expressing the approximate profile in terms of ς_3 will be an alternative for solving analytically the problem. In such case, the profile has the following form :

$$u=(u_i-\varsigma_3)e^{\frac{\kappa}{k}\frac{s}{u_i-\varsigma_3}\frac{ds}{dt}}+\varsigma_3$$

and its substitution into the refined integral equation (11) leads to a similar ODE as equation (20); that is :

$$\frac{d\varsigma_3}{dt}=\frac{4\left(1-e^{\frac{\psi}{2\varsigma_3}}\right)\varsigma_3-2\psi}{\psi_1-\psi_2}(\varsigma_3+1) \qquad (27)$$

where

$$\psi_1=6-4\sqrt{2}+(2-\sqrt{2}+\psi)t+8\varsigma_3(3\varsigma_3-\psi)$$

$$\psi_2=4\varsigma_3(6\varsigma_3+\psi)e^{\frac{-\psi}{2\varsigma_3}}$$

It is clear that the solution is $\varsigma_3=-1$ since $\varsigma_3+1=0$ at $t=0$ (from the third initial condition of Eqs. (24) and the denominator is not zero as well. The exact solution is then recovered.

It is worthy to note that both previous examples are nothing more than particular cases of the problem solved exactly by Stefan himself using the wave variable. The problem is simplified since the moving front velocity, v, is constant. Therefore, the solution reads as follows:

$$u=u_i-\frac{\alpha\kappa}{k}\left(1-e^{\frac{v}{\alpha}(v(t+t_0)-x)}\right)$$

$$s(t)=v(t+t_0)$$

t_0 stands for the initial time. Examples 1 and 2 are obtained by considering the values of (v, t_0) as (1/10,10) and ($\sqrt{2}/2, 2-\sqrt{2}$) respectively. It should be stressed here that, according to Myers and Mitchell [3], the simple exponential profile, used here (Eq. 15) would lead to less accurate solutions compared to those obtained by the Gaussian profile as the one used by the authors; that is:

$$u=\varsigma_1+\varsigma_2\frac{x}{s}e^{\varsigma_3\left(\frac{x}{s}\right)^2}$$

One should recall that both profiles were used, in conjunction with the heat balance integral method, by Mosally *et al.* [19] for the Stefan problem with fixed temperature boundary. Indeed, in this case the Gaussian profile gave more accurate results since it is suggested by the expansion of the exact solution. On the other hand, Myers and Mitchell [9] pointed out that integral methods does not work well whenever the fixed boundary is time-dependent. The main result of the above analysis showed that this is not always the case.

III.3 Example 3

The following example is the one investigated by Johansson *et al.* [8]. The mathematical model is described by the diffusion equation (1) with $\alpha=1$, $f(x,t)=0$ and submitted to boundary conditions expressed in general form (Eqs.6,7) by :

$$h_1(t)=0 \qquad (28)$$

$$h_2(t) = \sqrt{3-2t} \tag{29}$$

The functions describing the time-evolution of the moving front and the initial temperature distribution are respectively

$$s(t) = 2 - \sqrt{3-2t} \tag{30}$$

$$\phi(x) = -\frac{1}{2}\left(x^2 - 4x + 1\right) \tag{31}$$

The authors proposed to recover the heat flux and the temperature along the fixed boundary $x = 0$ at any time t in the domain [0,1]. However, as it can be seen from the expression of the function $s(t)$, the moving boundary position increases from its initial value $s(t) = 2 - \sqrt{3}$ until it reaches the position $x = 2$ at $t = 1.5$. In the present study, the temperature distribution in the whole layer $x \in [0, s(t)]$ and during the entire period $t \in [0,1.5]$ is investigated.

It is worthwhile to repeat what was noticed by Johansson *et al.* [8], i.e., the problem is more realistic from the practical point of view since it involves a non-linear variation of the moving boundary $s(t)$ with time while Eq. (7), taking into account of Eqs. (29,30), is different from the classical Stefan condition (Eq. 5). In the present case, the refined integral equation (11) reads :

$$\frac{d}{dt}\int_0^s xudx + \alpha\left(u|_0 + sh_2\right) + \left(\alpha - s\frac{ds}{dt}\right)h_1 = 0 \tag{32}$$

The initial distribution (Eq. 31) suggests the following quadratic polynomial form for the kernel function :

$$u = \varsigma_1 + \varsigma_2 x + \varsigma_3 x^2 \tag{33}$$

Imposing the boundary conditions (Eqs.6,7) determines the shape functions ς_1 and ς_2 in terms of ς_3

$$\varsigma_1 = u|_0 = h_1 + (s\varsigma_3 - h_2)s\,, \qquad \varsigma_2 = h_2 - 2s\varsigma_3$$

Examining $u(x,0)$ from Eq.(33) and the initial distribution (Eq.31) it appears that the shape functions have the following initial values

$$\varsigma_1(0) = -\frac{1}{2}\,,\quad \varsigma_2(0) = 0\,,\quad \varsigma_3(0) = -\frac{1}{2} \tag{34}$$

Then, setting α and taking into account Eqs.(28-30) and substituting the function u from Eq.(32) into the integral equation (32) to yield the following ODE :

$$\frac{d\varsigma_3}{dt} = \frac{4(\varsigma_3 + 1)\left(31 - 18\sqrt{3-2t} + 2\left(2\sqrt{3-2t} - 9\right)t\right)}{\psi_1 - 4t\psi_2} \tag{35}$$

where, this time

$$\psi_1 = 168 - 97\sqrt{3-2t}$$

$$\psi_2 = 40 - 15\sqrt{3-2t} + \left(\sqrt{3-2t} - 8\right)t$$

It is also clear that the initial condition $\varsigma_3(0) = -1/2$ is the solution of this first order ODE. It results that $\varsigma_1(t) = -t - 1/2$, $\varsigma_2(t) = 2$ and $\varsigma_3(t) = -1/2$ and consequently, the assumed profile (Eq.33) becomes

$$u = -t - \frac{1}{2} + 2x - \frac{1}{2}x^2 \tag{36}$$

expressing therefore the exact solution of the test example considered.

III.4 Example 4

Finally, the problem already solved using the heat balance integral method [1] and various explicit numerical schemes [5] is considered. It concerns the case where a heat source is applied to the active region enclosed between the surface exchange $x = 0$ and the moving boundary $x = s$. This test example is considered according to the following parameters : $\alpha = 1$, $k = 1$, $\kappa = 2$ and $u_i = 0$, $u_0 = 0$ and

$$f(x,t) = xe^t + 4 \tag{37}$$

$$\phi(x) = x(1-2x) \tag{38}$$

$$s_0 = 0.5 \tag{39}$$

The position of the moving boundary $s(t)$ is unknown at this stage and constitutes a part of the solution. The exact solution of the Stefan problem, as stated, reads

$$s(t) = 0.5e^t \tag{40}$$

$$u(x,t) = x\left(e^t - 2x\right) \tag{41}$$

Notice that in both previous studies [1,5], the solution of the problem was considered for $\alpha=1$, $k=1$, $\kappa=1$, $s_0=1$, $f(x,t)=xe^t+2$ and $\phi(x)=x(1-x)$. The initial temperature distribution suggests to consider the previous quadratic profile form (Eq.33) as the kernel approximation $u(x,t)$ in the integral equation (11). The profile (Eq.33) fulfilling the boundary conditions (Eqs.4,5,8) and the initial condition (Eq.38) leads to the following sets of equations :

$$\varsigma_1=u_0,\qquad \varsigma_2=\frac{2}{s}(u_i-u_0)+\frac{\kappa}{k}\frac{ds}{dt}\quad\text{and}$$

$$\varsigma_3=-\frac{1}{s^2}(u_i-u_0)-\frac{1}{s}\frac{\kappa}{k}\frac{ds}{dt} \tag{42}$$

$$\varsigma_1(0)=0,\ \varsigma_2(0)=1,\quad \varsigma_3(0)=-2 \tag{43}$$

respectively. The resulting profile, with the numerical values previously given for α, k, κ, u_i and u_0, can be expressed in terms of the active layer thickness as

$$u=\frac{ds^2}{dt}\frac{x}{s}\left(1-\frac{x}{s}\right) \tag{44}$$

Upon substituting the above distribution into the refined integral equation (11), the following ODE is obtained :

$$s^2\frac{d^2s}{dt^2}+2s\left(\frac{ds}{dt}\right)^2-\frac{3}{2}e^ts^2=12\left(s-\frac{ds}{dt}\right) \tag{45}$$

We notice that the above second order ODE exhibits a coefficient varying exponentially with time t. Then, it can be assumed that the time-evolution of the moving boundary involves the function e^t. In such case, the solution may be obtained analytically if we proceed by separating the terms of the ODE according to their order. Considering the equation containing only the terms of order 1, i.e. the terms of RHS of the ODE, yields $ds/dt=s$, and consequently $d^2s/dt^2=ds/dt$. Substituting this in the rest of the ODE leads to $s=e^t/2$. It is worth underlying that, even if the ODE is of a second order, its integration does not require any initial condition.

Also, Eq.(45) can be solved numerically. Then, besides Eq.(39), a second initial condition is required by the numerical integration of the above ODE. This concerns the initial velocity of the moving front which is deduced from the second equation of Eqs.(42); that is

$$\frac{ds}{dt}(0)=\frac{k}{\kappa}\varsigma_2(0)=\frac{1}{2} \qquad 46)$$

The numerical integration, according to the initial conditions (Eqs.39,46), shows that the moving boundary position coincides numerically with both its velocity and acceleration. Then, equating these three quantities and substituting into the ODE (Eq.45) yields $s=e^t/2$ which is the exact solution. Therefore the assumed approximate profile (Eq.44) turns out to be also the exact profile.

IV. CONCLUDING REMARKS

In this paper four examples of Stefan-like problems were investigated using the refined integral method [11,20]. In each problem, the corresponding given initial distribution of the dependent variable was used to define the form of the approximate profile to be substituted into the integral equation. As result, the technique has led to the corresponding exact solution in each benchmark test example and has been proven to be a simple and a particularly effective technique to solve 1-D nonlinear transient diffusion problems.\par

This study showed that the simple exponential profile leads to the exact solution. Such statement contradicts the remark raised up by Mitchell and Myers [3] claiming that this type of profile would always lead to solutions less accurate than those obtained by the Gaussian profile. Furthermore, this work exhibits that the integral method could be used despite a non uniform initial distribution for which, according to Gupta and Banik [2], the conventional integral methods would not give good results at all.

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[1] N. Sadoun, E.K. Si-Ahmed, P. Colinet and J. Legrand, *Thermal Science* **13**, 81 (2009).
[2] R. S. Gupta, N. Banik, *Computer Meth. Appl*. *Mech*. *Eng*. **67**, 211 (1988).
[3] S. L. Mitchell, T. G. Myers, *SIAM revue* **52**, 57 (2010).
[4] S. Mitchell, Appl. Math. Model. **38**, 4396 (2014).
[5] N. Sadoun, E. K. Si-Ahmed, P. Colinet, J. Legrand, *Comptes Rendus Mecanique* **40**, 501 (2012).
[6] J. Crank, *Free and moving boundary problems*, (Clarendon Press, Oxford, 1984).
[7] I. M. Helbling, J. C. D. Ibarra, J. L. Luna, M. I. Cabrera, R. J. A. Grau, *J. Memb. Sci.* **350**, 10 (2010).
[8] B. T. Johansson, D. Lesnic, T. Reeve, *Appl. Math. Model*. **35**, 4367 (2011).
[9] T. G. Myers, *Int. J. Heat Mass Transfer* **53**,1119 (2010).
[10] N. Sadoun, E. K. Si-Ahmed, in: D. P. Lewis R. W. (Ed.), *Numerical Methods in Thermal Problems* **9** Pineridge Press, Swansea (1995) 843.
[11] N. Sadoun, E. K. Si-Ahmed, P. Colinet, *Appl. Math. Model.* **30**,531 (2006).
[12] T. R. Goodman, *J. Heat Transfer* **80**, 335 (1958).
[13] M. S. El-Genk, A. W. Cronenberg, *Int. J. Heat Mass Transfer* **22**,167 (1979).
[14] N. Sadoun, P. Colinet, E. K. Si-Ahmed, in: *Abstracts of the International Conference on Multiscale Complex Fluid Flows and Interfacial Phenomena* (MULTIFLOW 2010), Brussels, Belgium (2010).
[15] S. Slota, *Int. Commun. Heat Mass Transfer* **37**, 587 (2010).
[16] E. Hetmaniok, I. Nowak, S. Slota, R. Witula, *Int. Commun. Heat Mass Transfer* **39**, 30 (2012).
[17] D. A. Murio, *Comput. Math. Appl*. **24**, 45(1992).
[18] C. S. Liu, *Int. J. Heat Mass Transfer* **54**, 1941 (2011).
[19] F. Mosally, A. S. Wood, A. Al-Faid, *Appl. Math. Comput.* **130**,87 (2002).
[20] N. Sadoun, *Thèse de Magister*, Fac.de Physique, USTHB (2006).